On endomorphisms of the Cuntz algebra which preserve the canonical UHF-subalgebra, II

Hayashi, Tomohiro; Hong, Jeong Hee; Szymanski, Wojciech

Published in:
Journal of Functional Analysis

DOI:
10.1016/j.jfa.2016.07.004

Publication date:
2017

Document version
Early version, also known as pre-print

Citation for pulished version (APA):
Hayashi, T., Hong, J. H., & Szymanski, W. (2017). On endomorphisms of the Cuntz algebra which preserve the canonical UHF-subalgebra, II. Journal of Functional Analysis, 272(2), 759-775. DOI: 10.1016/j.jfa.2016.07.004

ON ENDOMORPHISMS OF THE CUNTZ ALGEBRA WHICH PRESERVE THE CANONICAL UHF-SUBALGEBRA, II

TOMOHIRO HAYASHI, JEONG HEE HONG, AND WOJCIECH SZYMAŃSKI

Abstract. It was shown recently by Conti, Rørdam and Szymański that there exist endomorphisms λ_u of the Cuntz algebra $\mathcal{O}_n$ such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ but $u \notin \mathcal{F}_n$, and a question was raised if for such a u there must always exist a unitary $v \in \mathcal{F}_n$ with $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$. In the present paper, we answer this question to the negative. To this end, we analyze the structure of such endomorphisms λ_u for which the relative commutant $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ is finite dimensional.

1. Introduction and preliminaries

This paper is devoted to continuation of the line of investigation of exotic endomorphisms of the Cuntz algebras initiated in [4]. Our main result is solution of a question raised therein, see below for details. Our startegy is based on a detailed analysis of such endomorphisms λ_u of $\mathcal{O}_n$ that globally preserve the core UHF subagebra $\mathcal{F}_n$ and have finite dimensional relative commutant $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$, and builds on the earlier results in this direction obtained in [10].

The Cuntz algebra $\mathcal{O}_n$, $n \geq 2$, is the C^*-algebra generated by isometries $S_1, \ldots, S_n$ satisfying $\sum_{i=1}^n S_i S_i{}^* = 1$. It is a purely infinite, simple C^*-algebra, independent of the choice of generating isometries, [7]. We denote by W_n^k the set of k-tuples $\mu = (\mu_1, \ldots, \mu_k)$ with $\mu_m \in \{1, \ldots, n\}$, and by W_n the union $\cup_{k=0}^{\infty} W_n^k$, where $W_n^0 = \{0\}$. If $\mu \in W_n^k$ then $|\mu| = k$ is the length of μ. If $\mu = (\mu_1, \ldots, \mu_k) \in W_n$ then $S_\mu = S_{\mu_1} \ldots S_{\mu_k}$ ($S_0 = 1$ by convention) is an isometry in $\mathcal{O}_n$. Every word in $\{S_i, S_i^* \mid i = 1, \ldots, n\}$ can be uniquely expressed as $S_\mu S_\nu^*$, for $\mu, \nu \in W_n$ [7, Lemma 1.3].

The gauge action γ of the circle group $\mathbb{T}$ on $\mathcal{O}_n$ is defined by $\gamma_z(S_i) = zS_i$, $z \in \mathbb{T}$. Let $\mathcal{F}_n$ be the fixed point algebra of γ. Denote $\mathcal{F}_n^{(k)} := \mathrm{span}\{S_\mu S_\nu^* \mid \mu, \nu \in W_n^k\}$. Then $\mathcal{F}_n$ is generated by $\mathcal{F}_n^{(k)}$, $k = 1, 2, \ldots$, and each $\mathcal{F}_n^{(k)}$ is isomorphic to the matrix algebra $M_{n^k}(\mathbb{C})$. Thus $\mathcal{F}_n$ is isomorphic to the UHF-algebra of type n^∞, and hence it has a

Date: March 30, 2016.

T. Hayashi was supported by JSPS KAKENHI Grant No. 25400109. J. H. Hong was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant No. 2012R1A1A2039991). W. Szymański was partially supported by the FNU Research Project 'Operator algebras, dynamical systems and quantum information theory' (2013–2015), the Villum Fonden Research Grant 'Local and global structures of groups and their algebras' (2014–2018), and by the Mittag-Leffler Institute during his stay there in January-February, 2016.

unique tracial state τ. There exists a faithful conditional expectation $E : \mathcal{O}_n \to \mathcal{F}_n$, defined by integration with respect to the Haar measure on $\mathbb{T}$ as

$$E(x) = \int_{\mathbb{T}} \gamma_z(x) dz.$$

For each $k \in \mathbb{Z}$ we denote by $\mathcal{O}_n^{(k)}$ the corresponding spectral subspace for γ in $\mathcal{O}_n$,

$$\mathcal{O}_n^{(k)} := \{x \in \mathcal{O}_n \mid \gamma_z(x) = z^k, \ \forall z \in \mathbb{T}\}.$$

Thus, in particular, $\mathcal{O}_n^{(0)} = \mathcal{F}_n$.

The C^*-subalgebra of $\mathcal{O}_n$ generated by projections $P_\mu := S_\mu S_\mu^*$, $\mu \in W_n$, is a MASA (maximal abelian subalgebra) in $\mathcal{O}_n$. We call it the diagonal and denote $\mathcal{D}_n$, also writing $\mathcal{D}_n^k$ for $\mathcal{D}_n \cap \mathcal{F}_n^{(k)}$.

The canonical shift endomorphism $\varphi : \mathcal{O}_n \to \mathcal{O}_n$ is defined by

$$\varphi(x) = \sum_{i=1}^{n} S_i x S_i^*.$$

It is easy to see that $S_i x = \varphi(x) S_i$ and $x S_i{}^* = S_i{}^* \varphi(x)$ for all $x \in \mathcal{O}_n$.

As shown by Cuntz in [8], there exists a bijective correspondence between unitaries in $\mathcal{O}_n$ (whose collection is denoted $\mathcal{U}(\mathcal{O}_n)$) and unital $*$-endomorphisms of $\mathcal{O}_n$, determined by

$$\lambda_u(S_i) = u S_i, \quad i = 1, \ldots, n.$$

We have $\mathrm{Ad}(u) = \lambda_{u\varphi(u^*)}$ for all $u \in \mathcal{U}(\mathcal{O}_n)$. If $u \in \mathcal{U}(\mathcal{O}_n)$ then for each positive integer k we denote

$$u_k = u\varphi(u) \cdots \varphi^{k-1}(u). \tag{1}$$

Here $\varphi^0 = \mathrm{id}$, and we agree that u_k^* stands for $(u_k)^*$. If α and β are multi-indices of length k and m, respectively, then $\lambda_u(S_\alpha S_\beta^*) = u_k S_\alpha S_\beta^* u_m^*$.

The Cuntz correspondence between unitaries and endomorphisms of $\mathcal{O}_n$ provides a very efficient tool for investigations of the latter. In this note, we continue the study (by several authors) of those unital endomorphisms which globally preserve the UHF-subalgebra $\mathcal{F}_n$. For example, such endomorphisms were analyzed from the point of view of the Jones-Kosaki-Watatani index theory in [12] and [3], and in connection with Hopf algebra actions in [9] and [13]. More recently, interesting combinatorial approaches to the study of permutative endomorphisms of this type have been found (e.g. see [6], [2], and a survey article [1]).

It was observed by Cuntz in his groundbreaking paper [8] that an *automorphism* λ_u globally preserves $\mathcal{F}_n$ if and only if $u \in \mathcal{F}_n$. The situation is more complex with *proper endomorphisms*. Clearly, $u \in \mathcal{F}_n$ implies $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$, [8], but the question if the converse is true remained open until very recently. Indeed, it was shown in [4] that there exist unitaries u in $\mathcal{O}_n \setminus \mathcal{F}_n$ such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$. All such examples found therein were of the form $u = wv$ with $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and $v \in \mathcal{F}_n$. In such a case, we also have $\lambda_u(x) = \lambda_v(x)$ for all $x \in \mathcal{F}_n$. Thus a natural question arises if such a factorization of u is always possible whenever $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ (cf. [1, Problem 5.3]).

Some progress towards answering this question has been made recently in [10] and [11]. The main purpose of the present paper is to develop definite methods for analyzing endomorphisms λ_u of $\mathcal{O}_n$ satifying $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and an additional condition that the relative commutant $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ be finite dimensional. In particular, we give a verifiable criterion for determining if the aforementioned decomposition is possible, Corollary 3.4. Based on this criterion, in Section 3 we give an explicit example of a unitary $u \in \mathcal{O}_2$ such that $\lambda_u(\mathcal{F}_2) \subseteq \mathcal{F}_2$ and $\dim \lambda_u(\mathcal{F}_2)' \cap \mathcal{F}_2 < \infty$ but there is no unitary $v \in \mathcal{F}_2$ such that $\lambda_u|_{\mathcal{F}_2} = \lambda_v|_{\mathcal{F}_2}$, see Example 3.6. In this way, we answer to the negative the question raised in [4] and [1].

2. The relative commutants

We begin by recording for future references a few simple facts, essentially contained in [4] and [10].

Proposition 2.1. *Let $u \in \mathcal{U}(\mathcal{O}_n)$. Then the following conditions are equivalent.*

(1) $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$,
(2) $\lambda_{\gamma_z(u)}|_{\mathcal{F}_n} = \lambda_u|_{\mathcal{F}_n}$ *for all* $z \in \mathbb{T}$,
(3) $u\gamma_z(u^*) \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ *for all* $z \in \mathbb{T}$.

Proof. Clearly, $\gamma_z \lambda_u \gamma_z^{-1} = \lambda_{\gamma_z(u)}$ for all $z \in \mathbb{T}$. Thus condition (2) above is equivalent to $\gamma_z \lambda_u|_{\mathcal{F}_n} = \lambda_u|_{\mathcal{F}_n}$ for all $z \in \mathbb{T}$. Obviously, this holds if and only if $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$. That is, (1) is equivalent to (2).

It is an immediate consequence of Proposition 2.1 and Proposition 4.7 from [4] that $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$ if and only if $vu^* \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. This gives (2) is equivalent to (3). □

Proposition 2.2. *If $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n = \mathbb{C}1$, then $u \in \mathcal{F}_n$.*

Proof. If $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n = \mathbb{C}1$, then $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n = \mathbb{C}1$ as well, [10, Theorem 1.1]. As shown in [4], this implies that $u \in \mathcal{F}_n$. □

Proposition 2.3. *Let u be a unitary in $\mathcal{O}_n$. Then $u = wv$ for some $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and a unitary $v \in \mathcal{F}_n$ if and only if there exists a unitary $y \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ such that $u\gamma_z(u^*) = y\gamma_z(y^*)$ for all $z \in \mathbb{T}$.*

Proof. If $u = wv$ for some $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and $v \in \mathcal{U}(\mathcal{F}_n)$, then $u\gamma_z(u^*) = w\gamma_z(w^*)$, and it suffices to put $y = w$.

Conversely, if there exists a unitary $y \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ such that $u\gamma_z(u^*) = y\gamma_z(y^*)$ for all $z \in \mathbb{T}$ then y^*u is fixed by all γ_z. Thus $y^*u \in \mathcal{F}_n$ and it suffices to put $w = y$ and $v = y^*u$. □

From now on, we make **a standing assumption** that $u \in \mathcal{U}(\mathcal{O}_n)$ is such that

$$\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n \text{ and } \dim\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty. \tag{2}$$

As shown in [10], assumption (2) above entails a number of important consequences, which we summarize as follows.

- We also have $\dim\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n < \infty$.
- There exists a unitary group $\{u_z\}_{z\in\mathbb{T}}$ in the center of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ such that $\mathrm{Ad}u_z(x) = \gamma_z(x)$ for all $x \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$.
- Minimal projections in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ are minimal in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ as well. Thus $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ contains a MASA consisting of projections in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$.

The proof of the following theorem is modelled after that of [10, Lemma 1.11].

Theorem 2.4. *Let $u \in \mathcal{U}(\mathcal{O}_n)$ be such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\dim\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$. Then there exist unitaries $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and $v \in \mathcal{O}_n$, and a unitary group $\{v_z\}_{z\in\mathbb{T}} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ satisfying $u = wv$ and $\gamma_z(v) = v_z v$ for all $z \in \mathbb{T}$.*

Proof. At first we note that $u\gamma_z(u^*)u_z$ is a unitary group in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. Indeed,

$$\begin{aligned}(u\gamma_{z_1}(u^*)u_{z_1})(u\gamma_{z_2}(u^*)u_{z_2}) &= u\gamma_{z_1}(u^*)(\mathrm{Ad}\, u_{z_1})(u)u_{z_1}\gamma_{z_2}(u^*)u_{z_2} \\ &= u\gamma_{z_1}(u^*)\gamma_{z_1}(u)\gamma_{z_1}(\gamma_{z_2}(u^*))u_{z_1}u_{z_2} = u\gamma_{z_1z_2}(u^*)u_{z_1z_2}.\end{aligned}$$

Since $\dim\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n < \infty$, this unitary group may be diagonalized. On the other hand, $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ contains a MASA composed of projections in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$. Thus, there exists a unitary $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ such that $y_z := w^*(u\gamma_z(u^*)u_z)w$ is a unitary group in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$. Since each u_z is in the center of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$, the unitary groups $\{y_z\}_{z\in\mathbb{T}}$ and $\{u_z\}_{z\in\mathbb{T}}$ commute.

Set $v_z := u_z y_z^*$, $z \in \mathbb{T}$, and $v := w^*u$. Then v_z is a unitary group in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ and

$$\gamma_z(v) = \gamma_z(w^*u) = u_z(u_z^*\gamma_z(w^*u)u^*w)w^*u = u_z y_z^* w^*u = v_z w^*u = v_z v.$$

for all $z \in \mathbb{T}$. This completes the proof. □

We keep the notation from Theorem 2.4, assuming that unitaries w, v and v_z have the properties described therein. Thus, in particular, $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$ by [4, Proposition 2.1]. Consequently, $\mathrm{Ad}v \circ \varphi$ is an automorphism of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$, by [4, Proposition 2.3 and Lemma 2.4].

Lemma 2.5. *With unitaries u, v, v_z and u_z as above, put*

$$X_z := (\mathrm{Ad}v \circ \varphi)(u_z)u_z^*v_z.$$

Then $\{X_z\}_{z\in\mathbb{T}}$ is a unitary group in the center of $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$, and we have

$$\gamma_z(v) = X_z u_z(\mathrm{Ad}v\circ\varphi)(u_z^*)v.$$

Proof. For each $x\in\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$, we see that

$$\begin{aligned} u_z(\mathrm{Ad}v\circ\varphi)(x)u_z^* &= \gamma_z(v\varphi(x)v^*) = \gamma_z(v)\gamma_z(\varphi(x))\gamma_z(v^*) = \gamma_z(v)\varphi(\gamma_z(x))\gamma_z(v)^* \\ &= v_z v\varphi(u_z x u_z^*)v^*v_z^* = v_z v\varphi(u_z)v^*v\varphi(x)v^*v\varphi(u_z^*)v^*v_z^* \\ &= v_z\,\mathrm{Ad}(v\varphi(u_z)v^*)((\mathrm{Ad}\,v\circ\varphi)(x))v_z^*. \end{aligned}$$

Hence, we have

$$\mathrm{Ad}(v_z^*u_z)((\mathrm{Ad}\,v\circ\varphi(x)) = \mathrm{Ad}((\mathrm{Ad}\,v\circ\varphi)(u_z))((\mathrm{Ad}\,v\circ\varphi)(x)).$$

Since $\mathrm{Ad}v\circ\varphi$ is an automorphism of $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$, this shows that

$$\mathrm{Ad}(v_z^*u_z) = \mathrm{Ad}((\mathrm{Ad}\,v\circ\varphi)(u_z)) \text{ on } \lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n. \tag{3}$$

Consequently, X_z belongs to the center of $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$.

Now, $\{u_z\}_{z\in\mathbb{T}}$ and $\{v_z\}_{z\in\mathbb{T}}$ are commuting unitary groups, and both commute with X_z, by the above argument. Therefore the unitary group $(\mathrm{Ad}\,v\circ\varphi)(u_z) = X_z u_z v_z^*$ commutes with both of them. Consequently, X_z being a product of three mutually commuting unitary groups itself is a unitary group.

The final claim of the lemma now follows from the fact that $\gamma_z(v) = v_z v$. □

Before proceeding further, we introduce the following notation. For $x\in\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$ and $k\in\mathbb{N}$, we set

$$x^{(k)} := x(\mathrm{Ad}v\circ\varphi)(x)(\mathrm{Ad}v\circ\varphi)^2(x)\cdots(\mathrm{Ad}v\circ\varphi)^{k-1}(x). \tag{4}$$

Lemma 2.6. *With unitaries u, v, v_z and u_z as above, and $x\in\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$, for all $g\in\mathcal{U}(\mathcal{O}_n)$, $z\in\mathbb{T}$ and $k\in\mathbb{N}$ we have the following identities.*

(i) $\gamma_z(v_k) = v_z^{(k)}v_k$,
(ii) $(\mathrm{Ad}g\circ\varphi)^k(x) = g_k\varphi^k(x)g_k^*$,
(iii) $v_z^{(k)} = X_z^{(k)}u_z(\mathrm{Ad}v\circ\varphi)^k(u_z^*)$,
(iv) $(gv)_k = g^{(k)}v_k$.

Proof. In all three cases, we proceed by induction on k.

Ad (i). Case $k=1$ is the identity $\gamma_z(v_1) = \gamma_z(v) = v_z v = v_z^{(1)}v_1$ from Theorem 2.4. For the inductive step, we calculate

$$\gamma_z(v_{k+1}) = \gamma_z(v_k\varphi^k(v)) = \gamma_z(v_k)\varphi^k(\gamma_z(v)) = v_z^{(k)}v_k\varphi^k(v_z)v_k^*v_k\varphi^k(v) = v_z^{(k+1)}v_{k+1}.$$

In this calculation we used identity (ii) of the present lemma, whose proof does not depend on (i).

Ad (ii). Case $k=1$ is clear. For the inductive step, we have

$$(\mathrm{Ad}\, g\circ\varphi)^{k+1} = (\mathrm{Ad}\, g\circ\varphi)(g_k\varphi^k(x)g_k^*) = g\varphi(g_k)\varphi^{k+1}(x)\varphi(g_k^*)g^* = g_{k+1}\varphi^{k+1}(x)g_{k+1}^*.$$

Ad (iii). Case $k=1$ is clear. For the inductive step, we see that

$$\begin{aligned} v_z^{(k+1)} &= v_z^{(k)}(\mathrm{Ad}\, v\circ\varphi)^k(v_z) = X_z^{(k)}u_z(\mathrm{Ad}v\circ\varphi)^k(u_z^*)(\mathrm{Ad}\, v\circ\varphi)^k(v_z) \\ &= X_z^{(k)}u_z(\mathrm{Ad}v\circ\varphi)^k(u_z^*v_z) = X_z^{(k)}u_z(\mathrm{Ad}v\circ\varphi)^k(X_z(\mathrm{Ad}\, v\circ\varphi)(u_z^*)) \\ &= X_z^{(k)}u_z(\mathrm{Ad}v\circ\varphi)^k(X_z)(\mathrm{Ad}\, v\circ\varphi)^{k+1}(u_z^*) = X_z^{(k+1)}u_z(\mathrm{Ad}v\circ\varphi)^{k+1}(u_z^*). \end{aligned}$$

Ad (iv). Case $k=1$ is clear. For the inductive step, we calculate using part (ii) above,

$$(gv)_{k+1} = (gv)_k\varphi^k(gv) = g^{(k)}v_k\varphi^k(gv) = g^{(k)}(v_k\varphi^k(g)v_k^*)v_k\varphi^k(v) = g^{(k+1)}v_{k+1},$$

and this completes the proof. □

The following lemma provides a key step in the proof of our second main result, Theorem 2.8, below. We continue keeping the notation of Theorem 2.4. Here we remark that since $v=w^*u$, $w\in\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$, and $\mathrm{Ad}\, u\circ\varphi$ is an automorphism of $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$, we see that $\mathrm{Ad}\, v\circ\varphi = \mathrm{Ad}\, w^*\circ(\mathrm{Ad}\, u\circ\varphi)$ is an automorphism of $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$ as well. We also note that for each positive integer k, $\{X_z^{(k)}\}_{z\in\mathbb{T}}$ is a unitary group in the center of $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$.

Lemma 2.7. *With unitaries u, v, v_z and u_z as above, there exist a positive integer k and a unitary $U\in\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$ such that*

$$(\mathrm{Ad}\, v\circ\varphi)^k(x) = \mathrm{Ad}\, U(x) \quad \textit{for all } x\in\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n.$$

Then $X_z^{(k)}=1$. Furthermore, for such U and k, we have $U^*v_k\in\mathcal{F}_n$.

Proof. Since $\mathrm{Ad}\, v\circ\varphi$ is an automorphism of a finite dimensional C^*-algebra $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$, its restricts to the center has finite order. Thus there exists a positive integer k and a unitary $U\in\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$ such that $(\mathrm{Ad}\, v\circ\varphi)^k = \mathrm{Ad}\, U$ on $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$. We claim that $U^*v_k\in\mathcal{F}_n$.

Indeed, by Lemma 2.6, for all $z\in\mathbb{T}$ we have

$$\gamma_z(v_k) = v_z^{(k)}v_k = X_z^{(k)}u_z(\mathrm{Ad}v\circ\varphi)^k(u_z^*)v_k = X_z^{(k)}u_zUu_z^*U^*v_k = X_z^{(k)}\gamma_z(U)U^*v_k,$$

and this yields

$$\gamma_z(U^*v_k) = X_z^{(k)}U^*v_k. \tag{5}$$

Since $\{X_z^{(k)}\}_{z\in\mathbb{T}}$ is a unitary group in the center of $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$, there exists a partition of unity $1=\sum_i p_i$ in $\mathcal{Z}(\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n)$ and integers k_i such that

$$X_z^{(k)} = \sum_i z^{k_i}p_i.$$

We have $(\mathrm{Ad}v\circ\varphi)^k(p_i) = Up_iU^* = p_i$ for all i. Combining this with part (ii) of Lemma 2.6, we get

$$v_k^* p_i v_k = \varphi^k(p_i). \tag{6}$$

We want to show that $k_i = 0$ for all i. Suppose for a moment this is not the case and let $k_i > 0$ for some i. We set $K := p_iU^*v_k(S_1^*)^{k_i}$. Since p_i being in $\mathcal{Z}(\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n)$ belongs to $\mathcal{F}_n$ as well, it follows from identity (5) above that

$$\gamma_z(K) = \gamma_z(p_iU^*v_k(S_1^*)^{k_i}) = p_iX_z^{(k)}U^*v_k\gamma_z((S_1^*)^{k_i}) = z^{k_i}p_iU^*v_k(z^{-k_i}(S_1^*)^{k_i}) = K.$$

Hence K belongs to $\mathcal{F}_n$. We have $KK^* = p_i$. On the other hand, using identity (6) we get

$$K^*K = S_1^{k_i}v_k^*p_iv_k(S_1^*)^{k_i} = S_1^{k_i}\varphi^k(p_i)(S_1^*)^{k_i} = \varphi^{k+k_i}(p_i)S_1^{k_i}(S_1^*)^{k_i}.$$

It easily follows that $\tau(KK^*) > \tau(K^*K)$, which is a contradiction. A similar argument applies in the case $k_i < 0$. Hence $k_i = 0$ for all i and thus $X_z^{(k)} = 1$. Now, identity (5) implies that U^*v_k is fixed by the gauge action and hence belongs to $\mathcal{F}_n$. □

Now, we are ready to prove the second main result of this paper.

Theorem 2.8. *Let $u \in \mathcal{U}(\mathcal{O}_n)$ be such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\dim\lambda_u(\mathcal{F}_n)'\cap\mathcal{F}_n < \infty$. Then there exist a positive integer k and unitaries $W \in \lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$ and $V \in \mathcal{F}_n$ such that $u_k = WV$.*

Proof. By Theorem 2.4 and Lemma 2.7, there exist unitaries $w, U \in \lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$, a unitary group $\{v_z\}_{z\in\mathbb{T}}$ in $\lambda_u(\mathcal{F}_n)'\cap\mathcal{F}_n$ and a positive integer k satisfying $u = wv$, $\gamma_z(v) = v_zv$, $U^*v_k \in \mathcal{F}_n$. By part (iv) of Lemma 2.6, we have $w^{(k)}v_k = u_k$. Thus to complete the proof, it suffices to put $W := w^{(k)}U$ and $V := U^*v_k$. □

It was observed in [4] (just above Remark 4.4) that if $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n = \mathbb{C}1$ then $u \in \mathcal{F}_n$. The following corollary gives a sharp strengthening of that result.

Corollary 2.9. *Let u be a unitary in $\mathcal{O}_n$. If $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$, $\dim\lambda_u(\mathcal{F}_n)'\cap\mathcal{F}_n < \infty$ and the automorphism $\mathrm{Ad}\,u\circ\varphi$ of $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$ is inner, then there exist a unitary $w \in \lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$ and a unitary $v \in \mathcal{F}_n$ such that $u = wv$, and hence also $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$. In particular, this is the case whenever $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$ is a factor.*

Remark 2.10. The assumption in Corollary 2.9 above that the automorphism $\mathrm{Ad}\,u\circ\varphi$ of $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$ be inner, is equivalent to demanding existence of a unitary g in the relative commutant $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$ such that

$$\lambda_{gu}(\mathcal{F}_n)'\cap\mathcal{O}_n = \lambda_{gu}(\mathcal{O}_n)'\cap\mathcal{O}_n.$$

Indeed, if $\mathrm{Ad}\,u\circ\varphi$ is inner then $\mathrm{Ad}\,gu\circ\varphi = \mathrm{id}$ for a suitable unitary g in $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$. Hence $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n = \lambda_{gu}(\mathcal{F}_n)'\cap\mathcal{O}_n = \lambda_{gu}(\mathcal{O}_n)'\cap\mathcal{O}_n$. Conversely, if $\lambda_{gu}(\mathcal{F}_n)'\cap\mathcal{O}_n =$

$\lambda_{gu}(\mathcal{O}_n)'\cap\mathcal{O}_n$ then $\mathrm{Ad}\,gu\circ\varphi=\mathrm{id}$ on $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n=\lambda_{gu}(\mathcal{F}_n)'\cap\mathcal{O}_n$, and hence $\mathrm{Ad}\,u\circ\varphi$ is inner. □

Remark 2.11. We remark that the implication in Corollary 2.9 above cannot be reversed. In fact, there exist unitaries $u\in\mathcal{F}_n$ such that $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$ is finite dimensional and the automorphism $\mathrm{Ad}\,u\circ\varphi$ is outer on $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$. For example, take

$$u=S_{22}S_{11}^*+S_{12}S_{22}^*+S_{11}S_{12}^*+P_{21},$$

a permutative unitary in $\mathcal{F}_2$. Then $\mathrm{Ad}\,u\circ\varphi$ is outer on $\lambda_u(\mathcal{F}_2)'\cap\mathcal{O}_2$. For otherwise let h be a unitary in $\lambda_u(\mathcal{F}_2)'\cap\mathcal{O}_2$ such that $\mathrm{Ad}\,u\circ\varphi=\mathrm{Ad}\,h$ on $\lambda_u(\mathcal{F}_2)'\cap\mathcal{O}_2$. Then $\mathrm{Ad}\,u\circ\varphi(h)=h$ and thus $h\in\lambda_u(\mathcal{O}_2)'\cap\mathcal{O}_2$. But it can be shown that λ_u is irreducible on $\mathcal{O}_2$ (e.g., see [5], where this endomorphism is denoted ρ_{142}), and hence $\lambda_u(\mathcal{O}_2)'\cap\mathcal{O}_2=\mathbb{C}1$. Thus h is a scalar and consequently $\mathrm{Ad}\,u\circ\varphi$ is identity on $\lambda_u(\mathcal{F}_2)'\cap\mathcal{O}_2$. This however is not the case, since one can calculate directly that $\mathrm{Ad}\,u\circ\varphi$ permutes P_1 and P_2, and both these projections are in $\lambda_u(\mathcal{F}_2)'\cap\mathcal{O}_2$. □

We want to elaborate a little bit the statement of Theorem 2.8 above. We continue keeping our standing assumption (2).

Lemma 2.12. *Let α be an automorphism of $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$ and let $k\in N$ be such that α^k acts trivially on $\mathcal{Z}(\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n)$. Then there exists a MASA D of $\lambda_u(\mathcal{F}_n)'\cap\mathcal{F}_n$ and a unitary g in $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$ such that*

(i) $(\mathrm{Ad}\,g\circ\alpha)^k=\mathrm{id}$, *and*

(ii) $(\mathrm{Ad}\,g\circ\alpha)(D)=D$.

Proof. Automorphism α permutes the finitely many minimal central projections of $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$. Write this permutation as a product of disjoint cycles. Clearly, it suffices to prove the lemma for each cycle separately. Thus we may simply assume that α acts transitively on minimal projections $p_1,p_2,\ldots,p_l$ in $\mathcal{Z}(\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n)$, so that $\alpha(p_i)=p_{i+1}$, with $p_{l+1}=p_1$. Let $\{e_{r,s}^{(i)}\}$ be matrix units of the full matrix algebra $p_i(\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n)$, such that all $e_{r,r}^{(i)}$ are in $\lambda_u(\mathcal{F}_n)'\cap\mathcal{F}_n$. Then $D:=\mathrm{span}\{e_{r,r}^{(i)}\}$ is a MASA in $\lambda_u(\mathcal{F}_n)'\cap\mathcal{F}_n$. Since $p_i(\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n)\cong p_{i+1}(\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n)$, we can find a unitary $g_i\in p_{i+1}(\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n)$ such that $(\mathrm{Ad}\,g_i\circ\alpha)(e_{r,s}^{(i)})=e_{r,s}^{(i+1)}$. Setting $g:=\sum_{i=1}^l g_i$ we obtain the desired result. □

Lemma 2.13. *Let $u\in\mathcal{U}(\mathcal{O}_n)$ be such that $\lambda_u(\mathcal{F}_n)\subseteq\mathcal{F}_n$ and $\dim\lambda_u(\mathcal{F}_n)'\cap\mathcal{F}_n<\infty$. Then there exist a positive integer k, a unitary $g\in\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$, and a unitary group $\{d_z\}_{z\in\mathbb{T}}\subseteq\lambda_u(\mathcal{F}_n)'\cap\mathcal{F}_n$ such that $(gv)_k\in\mathcal{F}_n$ and $\gamma_z(gv)=d_zgv$.*

Proof. Put $\alpha:=\mathrm{Ad}\,v\circ\varphi$, and let g and k be as in Lemma 2.12. Then we have

$$(\mathrm{Ad}\,gv\circ\varphi)^k=\mathrm{id},$$

and thus

$$\mathrm{Ad}\, v_k \circ \varphi^k = (\mathrm{Ad}\, v \circ \varphi)^k = \mathrm{Ad}(g^{(k)})^*$$

by parts (ii) and (iv) of Lemma 2.6. Then arguing as in the proof of Lemma 2.7 (with $g^{(k)*}$ playing the role of U), we get

$$(gv)_k = g^{(k)} v_k \in \mathcal{F}_n.$$

Now, let D be a MASA as in Lemma 2.12. For all $x \in D$ and $z \in \mathbb{T}$, we see that

$$\begin{aligned} gv\varphi(x)v^*g^* &= \gamma_z(gv\varphi(x)v^*g^*) = \gamma_z(g)v_z v\varphi(x)v^*v_z^*\gamma_z(g^*) \\ &= (\gamma_z(g)v_z g^*)(gv\varphi(x)v^*g^*)(\gamma_z(g)v_z g)^*, \end{aligned}$$

which implies that $\gamma_z(g)v_z g^*$ is in the commutant of MASA D, and hence in D itself. Set $d_z = \gamma_z(g)v_z g^*$, a unitary in D. Now, $d_z = u_z g u_z^* v_z g^*$ implies $u_z^* d_z = g(u_z^* v_z)g^*$. Since $\{u_z\}_{z\in\mathbb{T}}$ and $\{v_z\}_{z\in\mathbb{T}}$ are commuting unitary groups, so is $\{u_z^* d_z\}_{z\in\mathbb{T}}$, and consequently also is $\{d_z\}_{z\in\mathbb{T}}$. Finally, we see that $\gamma_z(gv) = \gamma_z(g)v_z v = d_z g v$. □

Now, we are ready to prove the following result.

Theorem 2.14. *Let $u \in \mathcal{U}(\mathcal{O}_n)$. If $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$, then there exists a unitary $W \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ satisfying the following.*

(i) *There exists a unitary group $\{d_z\}_{z\in\mathbb{T}} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ such that $\gamma_z(Wu) = d_z W u$ for all $z \in \mathbb{T}$.*

(ii) *There exists a positive integer k such that $(Wu)_k \in \mathcal{F}_n$.*

Proof. Let $u = wv$ be a factorization as in Theorem 2.4, and let $k \in \mathbb{N}$ and $g \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ be as in Lemma 2.13 above. Then setting $W := gw^*$ gives the claim. □

3. The criterion and examples

In this section, we give a dynamic characterization of those unitaries $u \in \mathcal{O}_n$ satisfying our standing assumptions which either belong to $\mathcal{F}_n$ (Theorem 3.2) or admit a unitary $v \in \mathcal{F}_n$ such that $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$ (Corollary 3.4). Before proving these results, we still need one technical lemma about the structure of the relative commutants. We keep our standing assumptions (2).

Lemma 3.1. *There exist a unitary group $\{q_z\}_{z\in\mathbb{T}}$ in $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$ such that*

$$X_z = q_z(\mathrm{Ad}\, v \circ \varphi)(q_z^*).$$

Proof. Since $\mathrm{Ad}\, v \circ \varphi$ restricts to an automorphism of $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$, there exist minimal projections $p_i^{(j)}$, $j = 1, \ldots, N$, $i = 1, \ldots, n_j$, in $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$ such that

$$\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) = \bigoplus_{j=1}^{N} \bigoplus_{i=1}^{n_j} p_i^{(j)}$$

and

$$(\mathrm{Ad}\, v \circ \varphi)(p_i^{(j)}) = p_{i+1}^{(j)} \text{ for } i < n_j, \text{ and } (\mathrm{Ad}\, v \circ \varphi)(p_{n_i}^{(j)}) = p_1^{(j)}.$$

Then X_z from Lemma 2.5 can be written as

$$X_z = \sum_{j=1}^{N} \sum_{i=1}^{n_j} z^{m_i^{(j)}} p_i^{(j)},$$

for some $m_i^{(j)} \in \mathbb{N}$. Now, let $k \in \mathbb{N}$ be such that $\mathrm{Ad}\, v \circ \varphi$ is an inner automorphism of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. Then

$$X_z^{(k)} = X_z(\mathrm{Ad}\, v \circ \varphi)(X_z)(\mathrm{Ad}\, v \circ \varphi)^2(X_z) \ldots (\mathrm{Ad}\, v \circ \varphi)^{k-1}(X_z) = 1$$

by Lemma 2.7. Since each n_j divides k, this implies that

$$\sum_{i=1}^{n_j} m_i^{(j)} = 0$$

for each $j = 1, \ldots, N$. Now, we want to define q_z as follows,

$$q_z = \sum_{j=1}^{N} \sum_{i=1}^{n_j} z^{r_i^{(j)}} p_i^{(j)},$$

for suitable chosen integers $r_i^{(j)}$, so that $X_z = q_z(\mathrm{Ad}\, v \circ \varphi)(q_z^*)$. To this end, it suffices to put

$$r_1^{(j)} ;= 0, \quad j = 1, \ldots, N,$$

$$r_k^{(j)} := \sum_{r=2}^{k} m_r^{(j)}, \quad j = 1, \ldots, N, \; k = 2, \ldots, n_j.$$

□

Theorem 3.2. *Let $u \in \mathcal{U}(\mathcal{O}_n)$ be such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$. Put $\alpha := \mathrm{Ad}\, u \circ \varphi$. If α satisfies the following two conditions:*

(i) *$\alpha(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n) = \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$, and*
(ii) *$\alpha|_{\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n}$ preserves the τ-trace,*

then $u \in \mathcal{F}_n$.

Proof. At first, we observe that there exists a unitary group $\{u'_z\}_{z\in\mathbb{T}}$ in $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n)$ such that $\mathrm{Ad}\, u'_z(x) = \gamma_z(x)$ for all $x \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and $\gamma_z(u) = u'_z \alpha({u'_z}^*)u$. Indeed, it suffices to put $u'_z := q_z u_z$, with q_z as in Lemma 3.1 above. Then $\alpha(u'_z) \in \mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n)$ by condition (i) of the theorem, and hence $\{u'_z \alpha({u'_z}^*)\}_{z\in\mathbb{T}}$ is a unitary group. Thus, $u'_z \alpha({u'_z}^*) = \sum z^{k_j} p_j$ for some integers k_j and a partition of unity by projections p_j from $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n)$.

Now, we claim that $p_j = 0$ whenever $k_j \neq 0$. To this end, suppose first that $k_j > 0$ for some index j, and put $R := p_{k_j} u (S_1^*)^{k_j}$. We have $\gamma_z(R) = R$ for all

$z \in \mathbb{T}$, and thus $R \in \mathcal{F}_n$. However, an easy calculation shows that $RR^* = p_{k_j}$ and $R^*R = \varphi^{k+1}(\alpha^{-1}(p_{k_j}))S_1^k(S_1^*)^k$. In view of condition (ii) of the theorem, this would imply $\tau(RR^*) \neq \tau(R^*R)$ if $p_j \neq 0$, a contradiction. Therefore $p_j = 0$ for all $k_j > 0$. A similar argument shows that $p_j = 0$ if $k_j < 0$.

Consequently, $u'_z\alpha({u'_z}^*) = 1$. But this gives $\gamma_z(u) = u$ for all $z \in \mathbb{T}$. Hence $u \in \mathcal{F}_n$ and the theorem is proved. $\square$

We note that Theorem 3.2 gives a necessary and sufficient condition for $u \in \mathcal{F}_n$, since the reverse implication is trivial. Likewise, Corollary 3.3 below, gives a necessary and sufficient condition for $u_k \in \mathcal{F}_n$.

Corollary 3.3. *Let $u \in \mathcal{U}(\mathcal{O}_n)$ be such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\dim\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$. Put $\alpha := (\mathrm{Ad}\, u \circ \varphi)^k$, for some positive integer k. If α satisfies the following two conditions:*

(i) *$\alpha(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n) = \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$, and*

(ii) *$\alpha|_{\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n}$ preserves the τ-trace,*

then $u_k \in \mathcal{F}_n$.

Now, we are ready to give the following decomposability criterion.

Corollary 3.4. *Let $u \in \mathcal{U}(\mathcal{O}_n)$ be such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\dim\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$. Put $\alpha := \mathrm{Ad}\, u \circ \varphi$. Then the following two conditions are equivalent:*

(1) *There exist unitaries $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and $v \in \mathcal{F}_n$ such that $u = wv$.*

(2) *For each minimal projection $p \in \mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$ there exists a τ-preserving isomorphism*

$$p(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n) \cong \alpha(p)(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n).$$

Now, we show how to construct examples of endomorphisms λ_u of $\mathcal{O}_n$ globally preserving the core UHF-subalgebra $\mathcal{F}_n$ but such that no unitary $v \in \mathcal{F}_n$ exists for which $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$.

To begin with, take two non-zero, orthogonal projections q_1, q_2 in $\mathcal{F}_n$ such that $\tau(q_2)/\tau(q_1) = n^r$ for some non-zero integer r. Let A_1 be a partial isometry in $\mathcal{O}_n^{(-r)}$ with domain projection $\varphi(q_1)$ and range projection q_2. Likewise, let A_2 be a partial isometry in $\mathcal{O}_n^{(r)}$ with domain projection $\varphi(q_2)$ and range projection q_1. Finally, let A_3 be a partial isometry in $\mathcal{F}_n$ with domain projection $1-\varphi(q_1)-\varphi(q_2)$ and range projection $1 - q_1 - q_2$. Put $u := A_1 + A_2 + A_3$. Then u is a unitary in $\mathcal{O}_n$ such that

$$\mathrm{Ad}\, u \circ \varphi(q_1) = q_2 \quad \text{and} \quad \mathrm{Ad}\, u \circ \varphi(q_2) = q_1. \tag{7}$$

Now, $u\gamma_z(u^*) = z^r q_1 + z^{-r} q_2 + 1 - q_1 - q_2$ belongs to $\mathrm{span}\{1, q_1, q_2\}$, and $\mathrm{span}\{1, q_1, q_2\} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ by [4, Proposition 2.3] and (7) above. Thus $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ by Proposition 2.1 above.

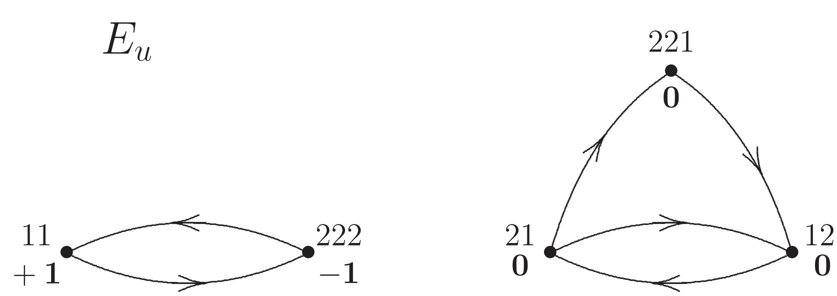
More generally, Let $1 = \sum q_j$ be a partition of unity by projections in $\mathcal{O}_n$. Let u be any unitary in $\mathcal{O}_n$ such that $\mathrm{Ad}\, u \circ \varphi$ permutes projections $\{q_j\}$ and for each j there is a $k_j \in \mathbb{Z}$ such that $q_j u \in \mathcal{O}_n^{(k_j)}$. Then $u\gamma_z(u^*) \in \mathrm{span}\{q_j\} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ for all $z \in \mathbb{T}$. This simple construction gives a large class of examples of unitaries $u \in \mathcal{O}_n \setminus \mathcal{F}_n$ such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$. However, to verify the conditions of Corollary 3.4 one needs more detailed information on the relative commutants $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. Exact determination of these relative commutants is rather difficult and does not seem possible in general, despite the identity from [4, Proposition 2.3]. However, it is quite doable in concrete cases.

Now, we illustrate the above discussion with two concrete examples in $\mathcal{O}_2$. In these examples, along with the main algebra $C^*(S_1, S_2) \cong \mathcal{O}_2$, we consider its other subalgebras, also isomorphic to $\mathcal{O}_2$. For example, if T_1, T_2 are isometries in $C^*(S_1, S_2)$ generating a copy of $\mathcal{O}_2$, then we use subscript T along with the standard notation to indicate that the object comes from $C^*(T_1, T_2)$ and its generators. Thus φ_T denotes the usual shift on $C^(T_1, T_2)$, that is a map $\varphi : C^*(T_1, T_2) \to C^*(T_1, T_2)$ such that $\varphi(x) = T_1 x T_1^* + T_2 x T_2^*$. Similarly, $\mathcal{D}_T$ denotes the diagonal MASA of $C^*(T_1, T_2)$, and so on. The proof of one technical lemma needed in Example 3.6 is given afterwards.

Example 3.5. Take $q_1 = P_{11}$, $q_2 = P_{222}$, and set

$$\begin{aligned} A_1 &= S_{2221}S_{111}^* + S_{2222}S_{211}^* \\ A_2 &= S_{111}S_{1222}^* + S_{112}S_{2222}^* \\ A_3 &= S_{1222}S_{2221}^* + S_{211}S_{112}^* + P_{121} + P_{1221} + P_{212} + P_{221} \end{aligned}$$

We note that unitary $u := A_1 + A_2 + A_3$ falls within the class of polynomial unitaries considered in [4, Section 5]. In particular, its graph E_u, as defined therein, admits the $\{-1, 0, +1\}$ labelling:



This labelled graph satisfies the path condition defined in [4, p. 616], and this is an alternative way of showing that $\lambda_u(\mathcal{F}_2) \subseteq \mathcal{F}_2$.

Now, we have $P_{11}\mathcal{O}_2P_{11} \cong \mathcal{O}_2 = C^*(T_1, T_2)$, under the isomorphism sending T_1 to $S_{111}S_{11}^*$ and T_2 to $S_{112}S_{11}^*$. Similarly, $P_{222}\mathcal{O}_2P_{222} \cong \mathcal{O}_2 = C^*(R_1, R_2)$, under the isomorphism sending R_1 to $S_{2221}S_{222}^*$ and R_2 to $S_{2222}S_{222}^*$. Then an easy calculation shows that

$$\operatorname{Ad} u \circ \varphi(T_j) = \varphi_R(R_j),$$
$$\operatorname{Ad} u \circ \varphi(R_j) = \varphi_T(T_j),$$

for $j = 1, 2$. Consequently, the restriction of $(\operatorname{Ad} u \circ \varphi)^2$ to $P_{11}\mathcal{O}_2P_{11}$ is conjugate to φ_R^2. Likewise, the restriction of $(\operatorname{Ad} u \circ \varphi)^2$ to $P_{222}\mathcal{O}_2P_{222}$ is conjugate to φ_T^2. This immediately implies

$$\lambda_u(\mathcal{F}_2)' \cap P_{11}\mathcal{O}_2P_{11} \subseteq \bigcap_{k=1}^{\infty}(\operatorname{Ad} u \circ \varphi)^{2k}(P_{11}\mathcal{O}_2P_{11}) = \mathbb{C}P_{11},$$
$$\lambda_u(\mathcal{F}_2)' \cap P_{222}\mathcal{O}_2P_{222} \subseteq \bigcap_{k=1}^{\infty}(\operatorname{Ad} u \circ \varphi)^{2k}(P_{222}\mathcal{O}_2P_{222}) = \mathbb{C}P_{222}.$$

That is, both P_{11} and P_{222} are minimal projections in $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$. One easily checks that $\operatorname{Ad} u \circ \varphi(S_{11}S_{222}^*) = S_{222}S_{11}^*$. Thus $S_{11}S_{222}^*$ is in $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$, and we see that $(P_{11} + P_{222})\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2(P_{11} + P_{222}) \cong \mathbb{M}_2(\mathbb{C})$. We remark that the restriction of $\operatorname{Ad} u \circ \varphi$ to $(P_{11} + P_{222})\mathcal{O}_2(P_{11} + P_{222})$ is conjugate to endomorphism ρ_{1342} from [5]. Let

$$w := S_{11}S_{222}^* + S_{222}S_{11}^* + 1 - P_{11} - P_{222}.$$

Then w is a unitary in $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$ such that $w^*u \in \mathcal{F}_2$. □

Example 3.6. Take $q_1 = P_1$, $q_2 = P_{21}$, and set

$$A_1 = S_{211}S_{21}^* + S_{2121}S_{112}^* + S_{2122}S_{111}^*,$$
$$A_2 = S_{12}S_{121}^* + S_{11}S_{221}^*,$$
$$A_3 = S_{221}S_{122}^* + P_{222}.$$

We put $u := A_1 + A_2 + A_3$. By construction, $\operatorname{Ad} u \circ \varphi(P_1) = P_{21}$ and also $\operatorname{Ad} u \circ \varphi(P_{21}) = P_1$. Hence $\operatorname{Ad} u \circ \varphi(P_{22}) = P_{22}$ as well.

We have $P_{22}C^*(S_1, S_2)P_{22} \cong \mathcal{O}_2 = C^*(R_1, R_2)$, under the identification of $S_{221}S_{22}^*$ with R_1 and $S_{222}S_{22}^*$ with R_2. This isomorphism yields a conjugation between the restriction of $\operatorname{Ad} u \circ \varphi$ to $P_{22}C^*(S_1, S_2)P_{22}$ and the shift φ_R. Consequently,

$$\lambda_u(\mathcal{F}_2)' \cap P_{22}C^*(S_1, S_2)P_{22} = \bigcap_{k=1}^{\infty}(\operatorname{Ad} u \circ \varphi)^k(P_{22}C^*(S_1, S_2)P_{22}) = \mathbb{C}P_{22}.$$

We have $P_1C^*(S_1, S_2)P_1 \cong \mathcal{O}_2 = C^*(T_1, T_2)$, under the identification of $S_{11}S_1^*$ with T_1 and $S_{12}S_1^*$ with T_2. This isomorphism carries the restriction of $(\operatorname{Ad} u \circ \varphi)^2$ to $P_1C^*(S_1, S_2)P_1$ to the endomorphism of $C^*(T_1, T_2)$ given as composition $\varphi_T \circ \psi_T$, where ψ_T is an endomorphism of $C^*(T_1, T_2)$ such that

$$\psi_T(x) = T_1xT_1^* + T_2(\operatorname{Ad} F_T(x))T_2^*,$$

where $F_T := T_2T_1^* + T_1T_2^*$. By Lemma 3.7, we have

$$\lambda_u(\mathcal{F}_2)' \cap P_1C^*(S_1,S_2)P_1 \subseteq \bigcap_{k=1}^{\infty}(\mathrm{Ad}\, u \circ \varphi)^{2k}(P_1C^*(S_1,S_2)P_1) = \mathbb{C}P_1.$$

We have $P_{21}C^*(S_1,S_2)P_{21} \cong \mathcal{O}_2 = C^*(V_1,V_2)$, under the identification of $S_{211}S_{21}^*$ with V_1 and $S_{212}S_{21}^*$ with V_2. This isomorphism carries the restriction of $(\mathrm{Ad}\, u \circ \varphi)^2$ to $P_{21}C^*(S_1,S_2)P_{21}$ to $\psi_V \circ \varphi_V$. An argument similar to that from Lemma 3.7 shows that $\lambda_u(\mathcal{F}_2)' \cap P_{21}C^*(S_1,S_2)P_{21} = \mathbb{C}P_{21}$. Alternatively, this also easily follows from the preceding argument and the fact that $\mathrm{Ad}\, u \circ \varphi(P_{21}) = P_1$.

In view of the above, either $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2 = \mathrm{span}\{P_1, P_{21}, P_{22}\} \cong \mathbb{C}^3$, or P_1 and P_{21} are equivalent in $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$. In the latter case, $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$ contains a subalgebra isomorphic to $M_2(\mathbb{C})$ which is invariant under $\mathrm{Ad}\, u \circ \varphi$ and has P_1 and P_{21} as its minimal projections. Suppose for a moment that this is the case. Then $\mathrm{Ad}\, u \circ \varphi$ restricts to a non-trivial automorphism of $M_2(\mathbb{C})$, by necessity inner. The implementing unitary matrix g is fixed by $\mathrm{Ad}\, u \circ \varphi$ and thus belongs to $\lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2$. Matrix g has both diagonal entries equal to 0. Multiplying g by a suitable scalar of modulus 1, we can find such g that is self-adjoint. Now we see that there is a unitary element x of $\mathcal{O}_2$ such that

$$g = S_{21}x^*S_1^* + S_1xS_{21}^* \in \lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2.$$

Now, writing $F := S_1S_2^* + S_2S_1^*$, we compute

$$\begin{aligned}\mathrm{Ad} u \circ \varphi(g) &= u(S_{11}xS_{121}^* + S_{121}x^*S_{11}^* + S_{21}xS_{221}^* + S_{221}x^*S_{21}^*)u^* \\ &= S_{212}FxS_{12}^* + S_{12}x^*FS_{212}^* + S_{211}xS_{11}^* + S_{11}x^*S_{211}^*,\end{aligned}$$

and hence we get

$$S_1xS_{21}^* + S_{21}x^*S_1^* = S_{212}FxS_{12}^* + S_{12}x^*F^*S_{212}^* + S_{211}xS_{11}^* + S_{11}x^*S_{211}^*.$$

Multiplying by S_1^* from the left-side and by S_{21} from the right-side, we obtain

$$x = S_2x^*FS_2^* + S_1x^*S_1^*. \tag{8}$$

Equation (8) implies $xS_1 = S_1x^*$ and $S_1^*x = x^*S_1^*$. These two combined then yield $(x + x^*)S_1 = S_1(x + x^*)$ and $(x - x^*)S_1 = -S_1(x - x^*)$. By [14, Proposition 4], both $x + x^*$ and $x - x^*$ are scalars, and thus so is x. This however contradicts (8).

Thus $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2 = \mathrm{span}\{P_1, P_{21}, P_{22}\}$ and since $\tau(P_1) \neq \tau(P_{21})$, we conclude from Corollary 3.4 that there are no unitaries $w \in \lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$ and $v \in \mathcal{F}_2$ such that $u = wv$.
□

Lemma 3.7. *Let ψ_T be an endomorphism of $C^*(T_1,T_2) \cong \mathcal{O}_2$ such that*

$$\psi_T(x) = T_1xT_1^* + T_2(\mathrm{Ad}\, F_T(x))T_2^*,$$

where $F_T := T_2T_1^* + T_1T_2^*$. *Then we have*

$$\bigcap_{k=1}^{\infty}(\varphi_T\psi_T)^k(C^*(T_1,T_2)) = \mathbb{C}1.$$

Proof. We note that

$$\varphi_T\psi_T(x) = T_{11}xT_{11}^* + T_{21}xT_{21}^* + T_{12}(\mathrm{Ad}\,F_T(x))T_{12}^* + T_{22}(\mathrm{Ad}\,F_T(x))T_{22}^*.$$

Also, we clearly have $F_TT_1 = T_2$ and $F_TT_2 = T_1$. Thus $(\varphi_T\psi_T)^k(x)$ may be written as a finite sum of elements of the form $T_\mu XT_\mu^*$ with $|\mu| = 2k$. This gives

$$\bigcap_{k=1}^{\infty}(\varphi_T\psi_T)^k(C^*(T_1,T_2)) \subseteq \mathcal{D}_T' \cap C^*(T_1,T_2) = \mathcal{D}_T.$$

For a positive ineger k, let

$$Q_k := \sum_{|\mu|=k-1} T_{\mu 1}T_{\mu 1}^*.$$

Then a straightforward induction on k shows that

$$Q_{2k}(\varphi_T\psi_T)^k(x) = Q_{2k}\varphi_T^{2k}(x) \tag{9}$$

for all $x \in C^*(T_1,T_2)$. Take a $d = d^* \in \mathcal{D}_T$ that belongs to $\bigcap_{k=1}^{\infty}(\varphi_T\psi_T)^k(C^*(T_1,T_2))$. Suppose d is not a constant multiple of 1. Then there exist $k \in \mathbb{N}$, $t \in \mathbb{R}$, $\epsilon > 0$ and $\mu,\nu \in W_2^{2k-1}$ such that

$$T_{\mu 1}T_{\mu 1}^*d \geq (t+\epsilon)T_{\mu 1}T_{\mu 1}^* \quad \text{and} \quad T_{\nu 1}T_{\nu 1}^*d \leq (t-\epsilon)T_{\mu 1}T_{\mu 1}^*.$$

Let $x = x^* \in \mathcal{D}_2$ be such that $d = (\varphi_T\psi_T)^k(x)$. Then $Q_{2k}d = Q_{2k}\varphi_T^{2k}(x)$. Since $T_{\mu 1}T_{\mu 1}^* \leq Q_{2k}$ and $T_{\nu 1}T_{\nu 1}^* \leq Q_{2k}$, we get

$$\begin{aligned} T_{\mu 1}xT_{\mu 1}^* &= T_{\mu 1}T_{\mu 1}^*Q_{2k}\varphi_T^{2k}(x) \geq (t+\epsilon)T_{\mu 1}T_{\mu 1}^*, \quad \text{and} \\ T_{\nu 1}xT_{\nu 1}^* &= T_{\nu 1}T_{\nu 1}^*Q_{2k}\varphi_T^{2k}(x) \leq (t-\epsilon)T_{\nu 1}T_{\nu 1}^*. \end{aligned}$$

This, however, is a contradiction. Indeed, since $T_{\mu 1}$ and $T_{\nu 1}$ are isometries, the above two inequalities would imply that both $x \geq (t+\epsilon)$ and $x \leq (t-\epsilon)$. Consequently,

$$\bigcap_{k=1}^{\infty}(\varphi_T\psi_T)^k(C^*(T_1,T_2)) = \mathbb{C}1,$$

as required. □

References

[1] R. Conti, J. H. Hong and W. Szymański, *Endomorphisms of the Cuntz algebras.* Banach Center Publ. **96** (2012), 81–97.

[2] R. Conti, J. H. Hong and W. Szymański, *The Weyl group of the Cuntz algebra.* Adv. Math. **231** (2012), 3147–3161.

[3] R. Conti and C. Pinzari, *Remarks on the index of endomorphisms of Cuntz algebras.* J. Funct. Anal. **142** (1996), 369–405.

[4] R. Conti, M. Rørdam and W. Szymański, *Endomorphisms of $\mathcal{O}_n$ which preserve the canonical UHF-subalgebra.* J. Funct. Anal. **259** (2010), 602–617.

[5] R. Conti and W. Szymański, *Computing the Jones index of quadratic permutation endomorphisms of $\mathcal{O}_2$.* J. Math. Phys. **50** (2009) 012705.

[6] R. Conti and W. Szymański, *Labeled trees and localized automorphisms of the Cuntz algebras.* Trans. Amer. Math. Soc. **363** (2011), 5847–5870.

[7] J. Cuntz, *Simple C^*-algebras generated by isometries.* Comm. Math. Phys. **57** (1977), 173–185.

[8] J. Cuntz, *Automorphisms of certain simple C^*-algebras.* in *Quantum fields-algebras-processes*, ed. L. Streit, Springer 1980.

[9] J. Cuntz, *Regular actions of Hopf algebras on the C^*-algebra generated by a Hilbert space.* Operator algebras, mathematical physics, and low-dimensional topology (Istanbul, 1991), 87–100, Res. Notes Math., 5, A. K. Peters, Wellesley, MA, 1993.

[10] T. Hayashi, *On normalizers of C^*-subalgebras in the Cuntz algebra $\mathcal{O}_n$.* J. Operator Theory **69** (2013), 525–533.

[11] T. Hayashi and W. Szymański, *On normalizers of C^*-subalgebras in the Cuntz algebra $\mathcal{O}_n$. II.* Banach J. Math. Anal. **9** (2015), 379–383.

[12] M. Izumi, *Subalgebras of infinite C^*–algebras with finite Watatani indices. I. Cuntz algebras.* Comm. Math. Phys. **155** (1993), 157–182.

[13] R. Longo, *A duality for Hopf algebras and for subfactors. I.* Comm. Math. Phys. **159** (1994), 133–150.

[14] K. Matsumoto and J. Tomiyama, *Outer automorphisms on Cuntz algebras*, Bull. London Math. Soc. **25** (1993), 64–66.

Nagoya Institute of Technology, Gokiso-cho, Showa-ku, Nagoya, Aichi, 466–8555, Japan

E-mail address: `hayashi.tomohiro@nitech.ac.jp`

Department of Data Information, Korea Maritime and Ocean University, Busan 606–791, South Korea

E-mail address: `hongjh@hhu.ac.kr`

Department of Mathematics and Computer Science, The University of Southern Denmark, Campusvej 55, DK–5230 Odense M, Denmark

E-mail address: `szymanski@imada.sdu.dk`